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Q.1. Define cosets and let $a \in G$ be arbitrary and let H be a subgroup of a group G. Then $Ha = H = aH \Leftrightarrow a \in H$.

Soln' Cosets: $\rightarrow$ Let H be a subgroup of a group $(G, *)$.

Let $a \in G$ be arbitrary. We define.

$$aH = \{ah : h \in H\} \text{ and } Ha = \{ha : h \in H\}$$

Then aH is called left coset of H in G generated by a, and Ha is called right coset of H in G generated by a.

Proof: Let H be a subgroup of G and let $a \in G$ be arbitrary.

Here, we have two steps arise.

Step I. We show that

$$Ha = H \Leftrightarrow a \in H$$

Let us first suppose $Ha = H$, to show $a \in H$

$$e \in H, a \in H \Rightarrow ea \in Ha \Rightarrow a \in Ha$$

$\Rightarrow a \in H$. For $H = Ha$

Now, let $a \in H$, to show $Ha = H$

Let $xa \in Ha \Rightarrow x \in H$
$\Rightarrow x \in H, a \in H$. For $a \in H$
$\Rightarrow xa \in H$ [$\because$ H is a subgroup]

Thus, any $xa \in Ha \Rightarrow xa \in H$

This prove that $Ha \subset H$ ——— (I)

$a \in H \Rightarrow a^{-1} \in H$ [$\because$ H is a subgroup]

For any $y \in H, a^{-1} \in H \Rightarrow ya^{-1} \in H$
$\Rightarrow (ya^{-1})a \in Ha$
$\Rightarrow y \in Ha$ for $ya^{-1}a = ye = y$

Thus any $y \in H \Rightarrow y \in Ha$
$\Rightarrow H \subset Ha$ ——— (II)

Now, from I and II, we get

$$H = Ha$$

Step II, To show $aH = H \Leftrightarrow a \in H$

We can prove Step II by making the parallel argument as in I. Proved

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Q.2. If a and b are arbitrary distinct elements of a group G and H is any subgroup of G, then

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$
$$aH = bH \Leftrightarrow b^{-1}a \in H.$$

Proof: Let a and b be arbitrary element of a group G such that $a \neq b$.

Let e be the identity of G $\Rightarrow e \in H$

Firstly, we shall show that

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$
$$Ha = Hb \Rightarrow (Ha)b^{-1} = (Hb)(b^{-1})$$
$$\Rightarrow H(ab^{-1}) = H(bb^{-1}) = He = H$$
$$\Rightarrow H(ab^{-1}) = H \Rightarrow ab^{-1} \in H$$
$$\Rightarrow H(ab^{-1}) = H$$

Conversely, $ab^{-1} \in H \Rightarrow H(ab^{-1})(b) = Hb$
$$\Rightarrow (Hab^{-1})(b) = Hb$$
$$\Rightarrow (Ha)(b^{-1}b) = Hb$$
$$\Rightarrow (Ha)e = Hb \Rightarrow Ha = Hb$$

Therefore, we have,
$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$

Similarly, we can prove that
$$aH = bH \Leftrightarrow b^{-1}a \in H.$$

Q.3. Any two left cosets of a subgroups are either disjoint or identical.

proof: Let aH and bH be any two left cosets of H. To show if aH and bH have an element in common, i.e. If $aH \cap bH$ is not the empty set, then they are identical, i.e. $aH = bH$.

Let $aH \cap bH \neq \phi$ and let c be any element of $aH \cap bH$, there there exist elements $h_1, h_2 \in H$ such that $c = ah_1$ and $c = bh_2$ it follows that
$$ah_1 = bh_2 \text{ so } a = bh_2(h_1)^{-1} \quad \text{—— (I)}$$

Now, Let ah be any element of aH. Then
$$ah = bh_2(h_1)^{-1}h.$$

Now, since H is a subgroup, $h_2(h_1)^{-1}h \in H$ and so $ah \in bH$.

This shows that every $ah \in aH$ is also in bH. Therefore
$$aH \subseteq bH.$$

Similarly, we can show that $bH \subseteq aH$.

Therefore, we have $aH = bH$.

Hence, we have shown that any two left cosets which are not disjoint are identical - proved.